C*-simplicity of Groups and Actions on Boundaries

Erik Babuschkin

November 8, 2022

Contents

1 The reduced C*-algebra of a group

Let G be a countable (discrete) group. Recall that the *complex group algebra* $\mathbb{C}[G]$ of G is the set of all finitely supported functions $f\colon G \to \mathbb{C}$ endowed with the operations

$$(f+g)(x) = f(x) + g(x), \text{ and}$$
$$(f * g)(x) = \sum_{y \in G} f(y) g(y^{-1}x).$$

Observation 1.1. Each $\varphi \in \mathbb{C}[G]$ acts as a bounded linear operator λ_φ on the Hilbert space

$$\ell^2(G) = \{f\colon G \to \mathbb{C} \mid \sum_{y \in G} |f(x)|^2 < \infty\}$$

by left convolution: $\lambda_\varphi f = \varphi * f$.

Definition 1.2. The *reduced C*-algebra* $C_r^*(G)$ of G is the norm closure of the image of the *left regular representation*

$$\begin{aligned} \lambda \colon \mathbb{C}[G] &\to \mathscr{B}(\ell^2(G)) \\ \varphi &\mapsto \lambda_\varphi. \end{aligned}$$

If $C_r^*(G)$ is simple, i. e. it has no non-trivial two-sided ideals, then G is said to be C*-simple.

Some conventions:

- G is always a (discrete) countable group with identity element e;
- K is a *compact* Hausdorff space;
- L is a Hausdorff space which is not necessarily compact.
- All group actions $G \curvearrowright K$ and $G \curvearrowright L$ are assumed to be continuous.

2 The theorem of Kalantar and Kennedy

Definition 2.1. We say that an action $G \curvearrowright L$ is

(i) *minimal*, if $\varnothing$ and L are the only invariant closed subsets of $G \curvearrowright L$;

(ii) *topologically free*, if for all $g \in G\backslash\{e\}$ the set

$$\mathrm{Fix}_L(g) = \{x \in X \mid g.x = x\}$$

has empty interior in L.

Theorem 2.2 (Kalantar–Kennedy, 2014)**.** *The following are equivalent.*

(i) *G is C*-simple;*

(ii) *There is an action $G \curvearrowright K$ which is*

  (a) *minimal;*

  (b) strongly proximal*;*

  (c) *topologically free.*

3 Strongly proximal actions

Let $\mathscr{M}(K)$ be the space of all Radon probability measures on K endowed with the *weak topology*. This is the coarsest topology such that all of the functions

$$T_f \colon \mu \mapsto \int_K f \,\mathrm{d}\mu$$

for $f \in C(K)$ are continuous.

Fact 3.1. The function δ sending $x \in K$ to its Dirac measure $\delta_x \in \mathscr{M}(K)$ is a topological embedding of K into $\mathscr{M}(K)$ called the *Dirac embedding*.

For compact Hausdorff spaces X and Y and a continuous function $f\colon X \to Y$, the *pushforward* of $\mu \in \mathscr{M}(X)$ along f is the measure $f_{\#}\mu \in \mathscr{M}(Y)$ defined by

$$(f_{\#}\mu)(A) = \mu(f^{-1}(A)) \quad (A \in \mathscr{B}(Y)).$$

The function

$$f_{\#}\colon \mathscr{M}(X) \to \mathscr{M}(Y)\mu \qquad \mapsto f_{\#}\mu$$

is called the *pushforward map* of f.

Given a group action $G \curvearrowright K$, we can define an action $G \curvearrowright \mathscr{M}(K)$ as follows: For each $g \in G$, let ρ_g denote the function $x \mapsto g.x$. Then for $\mu \in \mathscr{M}(K)$ we put $g.\mu = (\rho_g)_{\#}\mu$. The action of G on $\mathscr{M}(K)$ thus obtained is said to be *induced* by $G \curvearrowright K$. Observe that for $x \in K$ we have $g.\delta_x = \delta_{g.x}$, which means that δ is G-equivariant.

Definition 3.2. An action $G \curvearrowright K$ is *strongly proximal*, if the closure of each orbit of the induced action $G \curvearrowright \mathscr{M}(K)$ contains a Dirac measure.

Exercise 3.3. Can there be a strongly proximal action $G \curvearrowright [-1,1]$?

No! The subset $\{-1,1\}$ is invariant under every homeomorphism, as it is the unique subset of cardinality 2 with connected complement. Hence the measure $(\delta_{-1}+\delta_1)/2$ of $\mathscr{M}([-1,1])$ is a global fixed point of the induced action without being a Dirac measure.

4 (F)-boundaries

Definition 4.1. An action $G \curvearrowright K$ is called an (F)-boundary (action) for G, if it is minimal and strongly proximal.

With this terminology, the theorem of Kalantar and Kennedy states that G is C*-simple if and only if it has a topologically free (F)-boundary action. Note that this terminology is non-standard: We use the (F) to distinguish between different kinds of boundaries for groups.

Fact 4.2. (F)-boundaries have a rich structure. Here is a quick summary:

- The homeomorphism classes of (F)-boundaries for G form a complete lattice[1] with respect to the ordering

  $A \leq B$ if and only if "there is a G-equivariant continuous function $B \to A$".

- The least element of this lattice is always the class of the one-point space. The greatest element is the class of a space which is called the *Furstenberg boundary* of G, named after Harry Furstenberg who developped this boundary theory.

[1] Experts will notice the obliviousness of this statement regarding set-theoretic issues. In this case all issues can be resolved, as the minimality of the action gives an upper bound on the cardinality of an (F)-boundary. (Proof: Exercise)

5 Powers groups

Definition 5.1. G is said to be a *Powers group* if $G \neq \{e\}$ and for all finite subsets $F \subseteq G \backslash \{e\}$ and $N \in \mathbb{N}$, there is a partition $G = C \sqcup D$ and $\gamma_1, \gamma_2, \dots, \gamma_N \in G$ such that

1. $f.C \cap C = \varnothing$ for all $f \in F$;
2. $\gamma_i.D \cap \gamma_j.D = \varnothing$ whenever $i \neq j$.

Exercise 5.2 (For participants of the Random Walks School)**.** Prove that Powers groups admit a paradoxical decomposition and are therefore not amenable.

Let $F = \{f\}$ with an element $f \neq e$ of G and $N = 3$. We obtain a partition $G = C \sqcup D$ and elements $\gamma_1, \gamma_2, \gamma_3$ with the properties from the definition. Now put

$$A = f\gamma_1^{-1}\gamma_2 D \quad a = (f\gamma_1^{-1}\gamma_2)^{-1} \quad B = f\gamma_1^{-1}\gamma_3 D \quad b = (f\gamma_1^{-1}\gamma_3)^{-1}$$

to obtain mutually disjoint sets A, B and C with the property that

$$G = C \cup aA = C \cup bB.$$

Theorem 5.3 (Powers 1975, de la Harpe 1983)**.** *Every Powers group is C*-simple.*

Proof. See de la Harpe's survey. ■

Remark 5.4. The converse is not true! It can be shown that direct products of C*-simple groups are again C*-simple. However, non-trivial direct products of Powers groups are never Powers groups due to an argument by Promislow, which is reproduced in de la Harpes survey.

There is a criterion for determining Powers groups in terms of the dynamics of an action on a Hausdorff space, which will furnish us with many examples.

Definition 5.5. An action $G \curvearrowright L$ is *strongly faithful* if for all finite subsets $F \subseteq G \backslash \{e\}$ there is some $x \in L$ with $f.x \neq x$ for all $f \in F$.

Theorem 5.6 (de la Harpe, 1983)**.** *If $G \curvearrowright L$ is*

(i) *minimal;*

(ii) strongly hyperbolic;

(iii) *strongly faithful;*

then G is a Powers group.

Again, minimality and strong faithfulness control the size of the space L. We have not seen a definition for strong hyperbolicity yet. This will be given in the next section.

6 Strongly hyperbolic actions

Definition 6.1. We give the definition of strongly hyperbolic actions in three steps.

(i) A homeomorphism $\gamma\colon L \to L$ has *north-south dynamics* if it has two distinct fixed points r and a such that for any two neighborhoods R of r and A of a, there is some $n_0 \in \mathbb{N}$ such that

$$\gamma^n(L\backslash R) \subseteq A \quad \text{and} \quad \gamma^{-n}(L\backslash A) \subseteq R$$

for all $n \geq n_0$.

(ii) Two homeomorphisms γ and γ' from L to L with north-south dynamics are *transverse* if

$$\mathrm{Fix}_L(\gamma) \cap \mathrm{Fix}_L(\gamma') = \varnothing.$$

(iii) An action $G \curvearrowright L$ is *strongly hyperbolic* if G has infinitely many elements which act as mutually transverse homeomorphisms with north-south dynamics.

Exercise 6.2. Show that $G \curvearrowright L$ is strongly hyperbolic if there are *two* elements of G which act as transverse homeomorphisms with north-south dynamics.

Let γ_1 and γ_2 be two elements which act as transverse homeomorphisms with north-south dynamics. Let $\mathrm{Fix}_L(\gamma_1) = \{r, a\}$. The action of $\delta_n = \gamma_1^n\gamma_2\gamma_1^{-n}$ again has north-south dynamics. We want to show that there is an infinite subset $A \subseteq \mathbb{N}$ such that the elements of $\{\delta_n \mid n \in A\}$ act as mutually transverse homeomorphisms with north-south dynamics. Assume the contrary and derive a contradiction by showing that for each $i \in \mathbb{N}$, there can be at most one $j \neq i$ such that $P_i \cap P_j \neq \varnothing$, where

$$P_n = \mathrm{Fix}_L(\delta_n) = \{\gamma_1^n a, \gamma_1^n r\}$$

for all $n \in \mathbb{N}$.

7 (H)-boundaries

Definition 7.1. An action $G \curvearrowright L$ is called an *(H)-boundary* (action) for G if it is minimal and strongly hyperbolic.

Theorem 7.2. (*B.*) *Let $G \curvearrowright L$ be an (H)-boundary for G. Then*

(i) *$G \curvearrowright L$ is strongly faithful if and only if $G \curvearrowright L$ is strongly hyperbolic;*

(ii) *if L is compact and contains more than 2 elements, then L is an (F)-boundary for G;*

(iii) *if L is not compact then it is nowhere compact (i.e. every compact subset of L has empty interior).*

An open question: Does every Powers group have a (compact) (H)-boundary? This would complete the analogy between C*-simple groups and Powers groups and (F)-boundaries and (H)-boundaries.

Example 7.3. Some examples of Powers groups:

(*i*) Free products $G * H$ with $|G| \geq 2$ and $|H| \geq 3$ acting on the boundary of the corresponding Basse-Serre tree.

(*ii*) Non-soluble subgroups of $\mathrm{PSL}_2(\mathbb{R})$ acting on suitable subsets of $\partial \mathbb{H}$.

(*iii*) Torsion-free non-elementary Gromov-hyperbolic groups acting on their Gromov boundary.

(*iv*) Mapping class groups of surfaces of genus ≥ 1 acting on the boundary of the corresponding Teichmüller space.